

Even-dimensional topological gravity from Chern-Simons gravity.

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Abstract

It is shown that the topological action for gravity in $2n$ -dimensions can be obtained from the $(2n + 1)$ -dimensional Chern-Simons gravity genuinely invariant under the Poincaré group. The $2n$ -dimensional topological gravity is described by the dynamics of the boundary of a $(2n + 1)$ -dimensional Chern-Simons gravity theory with suitable boundary conditions.

The field ϕ^a , which is necessary to construct this type of topological gravity in even dimensions, is identified with the coset field associated with the non-linear realizations of the Poincaré group $ISO(d - 1, 1)$.

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I. INTRODUCTION

Twenty years ago A.H Chamseddine [1], [2] constructed topological actions for gravity in all dimensions. Chamseddine showed that the odd-dimensional theories are based on Chern-Simons forms with the gauge groups taken to be $ISO(2n, 1)$ or $SO(2n + 1, 1)$ or $SO(2n, 2)$ depending on the sign of the cosmological constant. The use of the Chern-Simons form was essential so as to have a gauge invariant action without constraints.

The even-dimensional theories use, in addition to the gauge fields, a scalar multiplet in the fundamental representation of the gauge group. For even-dimensional spaces there is no natural geometric candidate such as the Chern-Simons form. The wedge product of n of the field strengths can make the required $2n$ -form in a $2n$ -dimensional space-time. The natural gauge group is $ISO(2n - 1, 1)$ or $SO(2n, 1)$ or $SO(2n - 1, 2)$. To form a group invariant $2n$ -form, the n -product of the field strength is not enough, but will require in addition a scalar field ϕ^a in the fundamental representation.

It is the purpose of this paper to show that the topological action for gravity in $2n$ -dimensions can be obtained from the Chern-Simons gravity in $(2n + 1)$ -dimensions genuinely invariant under the Poincaré group. This Letter is organized as follows: In section 2 we shall review some aspects of (a) topological gravity [1], [2], (b) Lanczos-Lovelock gravity theory [6], [7], (c) the Stelle-West formalism [12] and of the Lanczos-Lovelock gravity theory genuinely invariant under the AdS group [9], [10], [11]. In section 3 it is shown that the topological action for gravity in $2n$ -dimensions, introduced in ref. [2] can be obtained from Chern-Simons gravity in $(2n + 1)$ -dimensions genuinely invariant under the Poincaré group. Section 4 contains some comments and the conclusions.

II. REVIEW ABOUT TOPOLOGICAL GRAVITY

A. Actions for topological gravity

In Ref. [1], [2] A.H Chamseddine constructed topological actions for gravity in all dimensions. For odd dimensions $d = 2n - 1$, the action is given by [1] $S_{2n+1} = k \int_{M_{2n+1}} \omega_{2n+1}$ where ω_{2n+1} is a Chern-Simons form given by [3] $\omega_{2n+1} = (n + 1) \int_0^1 dt \langle A (tdA + t^2 A^2)^n \rangle$ where $A = A^{ab} J_{ab}$ is the Lie algebra valued one form. Under a gauge transformation the gauge field A transform as $A^g = g^{-1} A g + g^{-1} dg$ and the Chern-Simons form transforms as

[3]

$$\omega_{2n+1}^g = \omega_{2n+1} + d\alpha_{2n} + (-1)^n \frac{n!(n+1)!}{(2n+1)!} \left\langle (g^{-1}dg)^{2n+1} \right\rangle. \quad (1)$$

Here α_{2n} is a $2n$ -form which is a function of A .

The even-dimensional theories use, in addition to the gauge fields, a scalar multiplet in the fundamental representation of the gauge group. To form a group invariant $2n$ -form, the n -product of the field strength is not enough, but will require in addition a scalar field ϕ^a in the fundamental representation. The $2n$ -dimensional action is then

$$I_{2n} = k \int_{M_{2n}} \varepsilon_{aa_2 \dots a_{2n+1}} \phi^a F^{a_2 a_3} \dots F^{a_{2n} a_{2n+1}}, \quad a = 0, 1, \dots, 2n \quad (2)$$

where $F^{ab} = dA^{ab} + A^{ac} A_c^b$.

This topological gravity has interesting applications, for example in 1+1-dimensions it allows one to describe Liouville's theory for gravity from a local Lagrangian.

B. Lovelock gravity theory

In Ref. [4], [5] was proved that the Lovelock lagrangian [6], [7] can be written as

$$S = \int \sum_{p=0}^{[d/2]} \alpha_p L^{(p)} \quad (3)$$

where α_p are arbitrary constants and $L^{(p)}$ is given by $L^{(p)} = \varepsilon_{a_1 a_2 \dots a_d} R^{a_1 a_2} \dots R^{a_{2p-1} a_{2p}} e^{a_{2p+1}} \dots e^{a_d}$ with $R^{ab} = d\omega^{ab} + \omega_c^a \omega^{cb}$.

In ref. [8] was shown that requiring that the equations of motion uniquely determine the dynamics for as many components of the independent fields as possible fixes the α_p coefficients in terms of the gravitational and cosmological constants. For $d = 2n$ the coefficients are $\alpha_p = \alpha_0 (2\gamma)^p \binom{n}{p}$, and the action (3) takes a Born-Infeld-like form. With these coefficients, the LL action is invariant only under local Lorentz rotations. For $d = 2n - 1$, the coefficients become

$$\alpha_p = \alpha_0 \frac{(2n-1)(2\gamma)^p}{(2n-2p-1)} \binom{n-1}{p}, \quad (4)$$

where $\alpha_0 = \frac{\kappa}{dl^{d-1}}$, $\gamma = -\text{sgn}(\Lambda) \frac{l^2}{2}$, and, for any dimension d , l is a length parameter related to the cosmological constant by $\Lambda = \pm(d-1)(d-2)/2l^2$. With these coefficients (4), the vielbein and the spin connection may be accommodated into a connection for the

AdS group, allowing for the lagrangian (3) to become the Chern-Simons form in $d = 2n + 1$ dimensions, whose exterior derivative is the Euler topological invariant in $d = 2n$ dimensions.

C. Lovelock gravity theory invariant under Poincaré group

In Refs. [9], [10], [11] it was shown that the Stelle-West formalism [12], which is an application of the theory of nonlinear realizations to gravity, permits constructing an action for Lanczos-Lovelock gravity theory genuinely invariant under the *AdS* group. In fact, a truly AdS-invariant action for even as well as for odd dimensions was constructed in ref. [10] using the Stelle-West formalism [12] for non-linear gauge theories. The action for this theory is

$$S_{\text{SW}}^{(d)} = \int \sum_{p=0}^{[D/2]} \alpha_p \varepsilon_{a_1 \dots a_d} R^{a_1 a_2} \dots R^{a_{2p-1} a_{2p}} V^{a_{2p+1}} \dots V^{a_d}, \quad (5)$$

where $V^a = \Omega_b^a (\cosh z) e^b + \Omega_b^a \left(\frac{\sinh z}{z} \right) D_\omega \phi^b$ and $\mathcal{R}^{ab} = dW^{ab} + W_c^a W^{cb}$ with $W^{ab} = \omega^{ab} + \frac{\sigma}{l^2} \left(\frac{\sinh z}{z} \right) e^c + \left(\frac{\cosh z - 1}{z^2} \right) D_\omega \phi^c (\phi^a \delta_c^b - \phi^b \delta_c^a)$ and $\Omega_b^a(u) \equiv u \delta_b^a + (1-u) \frac{\phi^a \phi_b}{\phi^2}$. Here ϕ^a corresponds to the so-called “(A)dS coordinate” which parametrizes the coset space $SO_{\hat{\eta}}(D+1)/SO_\eta(D)$, and $z = \phi/l$. This coordinate carries no dynamics, as any value that we pick for it is equivalent to a gauge choice breaking the symmetry from (A)dS down to the Lorentz group. This is best seen in the light that the equations of motion for the action (5) are the same as those for ordinary LL gravity, with e^a and ω^{ab} replaced by V^a and W^{ab} . The fields V^a and W^{ab} are called non-linear vielbein and spin connection, respectively, and they take up all the relevant information in the Stelle-West formalism.

From (5) we can see that, when one picks the physical gauge $\phi^a = 0$, the theory becomes indistinguishable from the usual one, and the AdS symmetry is broken down to the Lorentz group. However, a very interesting exception to this rule occurs in odd dimensions when the coefficients α_p (4) are chosen. In this case, and for any value of ϕ^a , it is possible to show that the Euler-Chern-Simons action written with e^a and ω^{ab} differs from that written with V^a and W^{ab} by a boundary term. As a matter of fact, the defining relation for the non-linear fields V^a and W^{ab} given in [12], represents a gauge transformation for the linear connection $\mathbf{A} = \frac{1}{2} i \omega^{ab} \mathbf{J}_{ab} - i e^a \mathbf{P}_a$, which can be written in the form $\mathbf{A} \rightarrow \tilde{\mathbf{A}} = g^{-1} (d + \mathbf{A}) g$, where $g = e^{-i \phi^a P_a}$ and $\tilde{\mathbf{A}} = \frac{1}{2} i W^{ab} \mathbf{J}_{ab} - i V^a \mathbf{P}_a$. This means that the linear and non-linear curvatures $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2$ and $\tilde{\mathbf{F}} = d\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^2$ are related by $\tilde{\mathbf{F}} = g^{-1} \mathbf{F} g$. Just

as the usual Euler-Chern-Simons lagrangian, the odd-dimensional non-linear lagrangian, with the special choice of coefficients given in eq. (4), satisfies $dL_{VW}^{(2n-1)} = \langle \tilde{\mathbf{F}}^n \rangle$, where $\langle \mathbf{J}_{a_1 a_2} \cdots \mathbf{J}_{a_{2n-3} a_{2n-2}} \mathbf{P}_{a_{2n-1}} \rangle = \frac{1}{l} \varepsilon_{a_1 \cdots a_{2n-1}}$. This implies that

$$dL_{VW}^{(2n-1)} = \langle \tilde{\mathbf{F}}^n \rangle = \langle \mathbf{F}^n \rangle = dL_{e\omega}^{(2n-1)}, \quad (6)$$

and hence we see that both lagrangians may locally differ only by a total derivative. The same arguments lead to the conclusion that, in general, any Chern-Simons lagrangian written with non-linear fields, which is genuinely invariant, differs from the usual one by a total derivative.

It is direct to show that in the limit $l \rightarrow \infty$ we obtain a Lovelock gravity theory genuinely invariant under the Poincaré group:

$$S_{SW}^{(d)} = k \int \varepsilon_{a_1 \cdots a_d} R^{a_1 a_2} \cdots R^{a_{d-2} a_{d-1}} V^{a_d}, \quad (7)$$

where now $V^a = e^a + D_\omega \phi^a = d\phi^a + \omega^a{}_b \phi^b + e^a$ and $R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb}$, with $W^{ab} = \omega^{ab}$, and ϕ^a corresponds to the so-called ‘‘Poincaré coordinate’’. The fields ϕ^a, e^a, ω^{ab} under local Poincaré translations change as $\delta\phi^a = -\rho^a$; $\delta e^a = \kappa^a{}_b e^b$; $\delta\omega^{ab} = -D\kappa^{ab}$.

III. TOPOLOGICAL GRAVITY FROM CHERN-SIMONS GRAVITY

In this section we show that the topological action for gravity in $2n$ -dimensions [2] can be obtained from $(2n+1)$ -dimensional Chern-Simons gravity genuinely invariant under the Poincaré group.

The Lanczos-Lovelock action genuinely invariant under the Poincaré group is given by

$$S = k \int \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}} V^{a_{2n+1}}. \quad (8)$$

Introducing the non-linear gauge fields V^a into (8) we obtain

$$\begin{aligned} S &= k \int_M \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \\ &+ k \int_M \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}} D\phi^{a_{2n+1}}, \\ S &= k \int_M \epsilon_{a_1 \cdots a_{2n+1}} R^{a_1 a_2} \cdots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \end{aligned} \quad (9)$$

$$+ k \int_M d [\epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \phi^{a_{2n+1}}], \quad (10)$$

where we have used the Bianchi identity $DR^{ab} = 0$. So that

$$\begin{aligned} S &= k \int_M \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \\ &+ k \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \phi^{a_{2n+1}}. \end{aligned} \quad (11)$$

This action differs by a boundary term from the usual Chern-Simons action for the Poincaré group written in terms of linear fields. This extra boundary term allows the action (8) to be genuinely gauge invariant and not only modulo boundary terms like the usual action in terms of linear gauge fields. This extra boundary term also changes the dynamic behavior of the Chern-Simons theory at the boundary of the manifold as we show below.

From (11) we can see that the whole dependence of the coset field ϕ^a is on the surface term and that the form of the surface term exactly coincides with the form of the even-dimensional action for even-dimensional topological gravity. However, the surface term cannot be directly considered as an action principle for the boundary, because the dynamics of the boundary is determined by the dynamics of the Bulk. We will show that, for solutions with suitable boundary conditions, it is possible to obtain the dynamics for the even-dimensional topological action. It must be noticed that the coset field ϕ^a , which is associated with a non-linear realization of the Poincaré group, appears in the action (11) in a geometrically natural form.

A. Invariance of the action

We show now that the action (11) is invariant under local Poincaré translations. In fact, under the transformations

$$\delta e^a = -D\rho^a; \quad \delta\phi^a = \rho^a \quad (12)$$

we obtain

$$\begin{aligned} \delta_{trasl} S &= -k \int_M \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} D\rho^{a_{2n+1}} \\ &+ k \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \rho^{a_{2n+1}} \\ &= -k \int_M d [\epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \rho^{a_{2n+1}}] \end{aligned} \quad (13)$$

$$+ k \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \rho^{a_{2n+1}}, \quad (14)$$

where we have used the Bianchi identity $DR^{ab} = 0$. This means that

$$\begin{aligned} \delta_{trasl} S &= -k \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \rho^{a_{2n+1}} \\ &+ k \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \rho^{a_{2n+1}} = 0. \end{aligned} \quad (15)$$

B. Equations of motion

The variations of the action (11) with respect to e^a, ω^{ab}, ϕ^a lead to

$$\begin{aligned} \delta S &= k \int_M n \epsilon_{a_1 \dots a_{2n+1}} D(\delta \omega^{a_1 a_2}) R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \\ &+ \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \delta e^{a_{2n+1}} \\ &+ k \int_{\partial M} n \epsilon_{a_1 \dots a_{2n+1}} D(\delta \omega^{a_1 a_2}) R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} \phi^{a_{2n+1}} \\ &+ \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \delta \phi^{a_{2n+1}} = 0. \end{aligned} \quad (16)$$

Integrating by parts we have

$$\begin{aligned} \delta S &= k \int_M n \epsilon_{a_1 \dots a_{2n+1}} \delta \omega^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} T^{a_{2n+1}} \\ &+ \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \delta e^{a_{2n+1}} \\ &+ k \int_{\partial M} n \epsilon_{a_1 \dots a_{2n+1}} \delta \omega^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \\ &+ n \epsilon_{a_1 \dots a_{2n+1}} \delta \omega^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} D \phi^{a_{2n+1}} \\ &+ k \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \delta \phi^{a_{2n+1}} = 0. \end{aligned} \quad (17)$$

From (17) we can see that by imposing the boundary conditions

$$\begin{aligned} &\int_{\partial M} n \epsilon_{a_1 \dots a_{2n+1}} \delta \omega^{a_1 a_2} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} [e^{a_{2n+1}} + D \phi^{a_{2n+1}}] \\ &+ \int_{\partial M} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \delta \phi^{a_{2n+1}} = 0 \end{aligned} \quad (18)$$

we obtain the following movement equations:

$$\epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} = 0 \quad (19)$$

$$\epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-3} a_{2n-2}} T^{a_{2n-1}} = 0, \quad (20)$$

which correspond to the usual Chern-Simons field equations.

From (18) we can see that the boundary condition associated with $\delta\phi^a$ is identically satisfied due to the validity of equation (19) on the boundary. Let's now consider a boundary condition of the Neumann type for the spin connection and, for the vielbein, a boundary condition of the type $e^a|_{\partial M} = 0$. This means that any solution of the field equations (19), (20) has a void vielbein on the boundary. Introducing the boundary condition $e^a|_{\partial M} = 0$ into (18) we obtain the following condition:

$$\epsilon_{a_1 \dots a_{2n+1}} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} D\phi^{a_{2n+1}}|_{\partial M} = 0 \quad (21)$$

Therefore, the dynamics of the boundary will be characterized by the following set of equations:

$$\epsilon_{a_1 \dots a_{2n+1}} R^{a_3 a_4} \dots R^{a_{2n-1} a_{2n}} D\phi^{a_{2n+1}}|_{\partial M} = 0 \quad (22)$$

$$\epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}}|_{\partial M} = 0 \quad (23)$$

which can be obtained from the action principle

$$S^{(2n)} = k \int_{\partial M_{2n+1} = M_{2n}} \epsilon_{a_1 \dots a_{2n+1}} R^{a_1 a_2} \dots R^{a_{2n-1} a_{2n}} \phi^{a_{2n+1}}. \quad (24)$$

This action correspond to topological gravity of ref.[2].

The condition $e^a|_{\partial M} = 0$ can be imposed due to the fact that e^a is a part of a connection, namely $A = \frac{i}{2}\omega^{ab}J_{ab} - ie^a P_a$. To have vielbeins and therefore non-invertible metrics in the boundary of a manifold means that the boundary of a manifold defines a singularity in the metric sector of the theory. However, from the point of view of the gauge structure, this does not represent any singularity due to the fact that the vielbein can be annulled because it is part of a gauge connection.

This shows that configurations with non invertible vielbeins can play an important role in the structure of the theory. Configurations with singularities on the boundary are not new (ref. [13]). They correspond to natural configurations of the gauge theories for gravity.

Finally it is interesting to notice that now the geometric origin of the ϕ^a field is clear, due to the fact that this one is a coset field associated with non-linear realizations of the Poincaré group.

IV. COMMENTS

We have shown in this work that the topological action for gravity in $2n$ -dimensions, introduced in ref. [2], can be obtained from the Chern-Simons gravity in $(2n + 1)$ -dimensions genuinely invariant under the Poincaré group. The $2n$ -dimensional topological gravity is described by the dynamics of the boundary of the $(2n + 1)$ Chern-Simons gravity theory with suitable boundary conditions. The boundary of the manifold defines a singular hypersurface in the metric sector of the theory. The singularity appears only when we consider configurations with metric invertibles. However, this singularity is not an intrinsic singularity of the theory due to the fact that the vielbein is a part of a gauge connection.

The dynamics on the boundary of a $(2n + 1)$ -dimensional manifold is described by the field equations of the topological gravity of ref. [2].

The field ϕ^a , which is necessary to construct this type of topological gravity in even dimensions [2], is identified by the coset field associated with non-linear realizations of the Poincare group $ISO(2n, 1)$. This shows a clear geometric interpretation of this field originally introduced "ad-hoc".

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[1] A. H. Chamseddine, Phys. Lett B **233** (1989) 291.

[2] A. H. Chamseddine, Nucl. Phys.B **346** (1990) 213.

- [3] M. Nakahara, Geometry, topology and physics,. Bristol, UK: IOP (2003) 573 p. (Graduate student series in physics).
- [4] B. Zumino, Phys. Rep. 137 (1986) 109.
- [5] C. Teitelboim and J. Zanelli, Class. and Quantum Grav. 4 (1987) L125.
- [6] C. Lanczos, Ann. Math. 39 (1938) 842.
- [7] D. Lovelock, J. Math. Phys. 12 (1971) 498.
- [8] R. Troncoso, J. Zanelli, Class. Quantum Grav. 17 (2000)4451.
- [9] P. Salgado, M. Cataldo, S. del Campo, Phys. Rev. D 66 (2002) 024013
- [10] F. Izaurieta, E. Rodriguez, P. Salgado, Phys. Lett.B **574** (2003) 283
- [11] F. Izaurieta, E. Rodriguez, P. Salgado, Phys. Lett.B **586** (2004) 397
- [12] K.S. Stelle, P.C. West, Phys. Rev. D21 (1980) 1466
- [13] G. Grignani, G. Nardelli, Phys. Rev. D45 (1992) 2719-2731